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Noncritical string correlators, finite-N matrix models and the vortex condensate

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Abstract: We carry out a systematic study of correlation functions of momentum modes in the Euclidean $c = 1$ string, as a function of the radius and to all orders in perturbation theory. We obtain simple explicit expressions for several classes of correlators in terms of special functions. The Normal Matrix Model is found to be a powerful computational tool that computes $c = 1$ string correlators even at finite N. This enables us to obtain a simple combinatoric formula for the $2n$-point function of unit momentum modes, which after T-duality determines the vortex condensate. We comment on possible applications of our results to T-duality at $c = 1$ and to the 2d black hole/vortex condensate problem.

Keywords: M(atrix) Theories, Bosonic Strings, Black Holes in String Theory.

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Contents

1. Introduction

The $c = 1$ string (an excellent review is ref. [1]) is a perturbatively consistent string theory in two spacetime dimensions. One of its attractive features is that it is solvable: from the powerful techniques of Matrix Quantum Mechanics (MQM), correlation functions of the momentum modes ("tachyons") can be determined to all orders in the string coupling (inverse cosmological constant). This holds true even in the Euclidean theory at finite radius R. Another feature is that its generalisation to the type 0 noncritical string, has similar properties in perturbation theory but is believed to also be non-perturbatively well-defined.

This makes the $c = 1$ string and its cousins a good laboratory to study various open questions in string theory. Two such questions that we would like to understand better in the noncritical context are the properties of string-scale black holes, and the nature of various dualities, including open-closed string duality [2 − 4]. Much work has been done on the former (some interesting recent studies can be found in refs. [5 − 7]), while the latter question has also yielded some important illuminations [8, 9]

It is known [15, 16] that basic properties of black holes in noncritical string theory are controlled by condensates of winding tachyons in the Euclidean-continued background. These are thermal tachyons: strings winding around the compact time direction. It would therefore be useful to know the correlators of winding modes in Euclidean noncritical string theory to all orders in the string coupling (and even nonperturbatively in the stable type-0 case) as a function of μ and R, where μ^{-1} is the inverse string coupling and R is the radius of the Euclidean direction (inverse temperature). From the matrix model point of view, winding modes are related to the nonsinglet sector of the model, in which the eigenvalue fermions are no longer free but mutually coupled [17, 18]. Computing correlators in this way is a harder task [19] and has raised some new puzzles involving leg factors which we will discuss in a later section. But one way to find the desired correlators is to assume that T-duality holds and perform it on the momentum correlators. This provides one of our motivations to study momentum correlators in the Euclidean theory in more explicit detail than has already been done.

As mentioned above, momentum correlators in the Euclidean $c = 1$ string are known in principle. They are summarised in the Toda hierarchy or W_∞ symmetries [20], or Hirota bilinear equations, or Normal Matrix Model (NMM) [21], all of which are supposed to be mutually equivalent. For the special case of self-dual radius $R = 1$ of the Euclidean time direction, they are encoded in a Kontsevich-Penner matrix model [22, 23] (see also [24, 25]). We will summarise some relevant information about these solutions below. But while all these formal solutions allow us to extract the perturbation series for any specific correlator after a sufficient amount of work, we do not have many explicit answers in terms of special functions depending on the radius R and inverse string coupling μ.

At finite radius, correlators have been computed mostly at tree-level (corresponding to the dispersionless limit of the Toda hierarchy) or to a few low orders in perturbation theory. For example, while the $2n$-point function of n unit winding modes and n anti-winding modes is known as a function of n and R at tree level [26, 16], an explicit expression for the same correlator to all orders in perturbation theory does not seem to exist in the literature.[1] To be more specific, denote by T_q the tachyons of momentum $q = n/R$, and by $\mathcal{T}_q$ the tachyons of n units of winding, where $q = nR$ is the value of $p_L = -p_R$ in vertex operator language. An explicit form is known for $\langle (T_{-1/R})^n (T_{1/R})^n \rangle$ at tree level. The T-dual of

[1] A differential equation for these correlators was written down in [16] together with an iterative solution to a few orders. Related work on Euclidean correlators can be found in refs. [27, 28].

this expression was used in ref. [16] to extract the critical behaviour of the Sine-Liouville theory defined by perturbing the original $c = 1$ string with $\mathcal{T}_{-R} + \mathcal{T}_R$ and then tuning the cosmological constant μ to zero. In particular, ref. [16] showed that a sensible theory exists after this tuning, but only when the radius of the Euclidean direction lies in the range $1 < R < 2$.

One would like to know the structure of this correlator to all string loop orders. Accordingly, in what follows we will study $\langle (T_{-1/R})^n (T_{1/R})^n \rangle$ in detail, and one of our main results will be a simple formula for this correlator as a function of μ and R for every n. We expect this to lead to a better understanding of the exponentiated correlator $\langle \exp(T_{-1/R} + T_{1/R}) \rangle$, which in turn is T-dual to the vortex condensate $\langle \exp(\mathcal{T}_{-R} + \mathcal{T}_R) \rangle$ that relates directly to Euclidean 2d black holes.

Another motivation for our work is to understand T-duality of the $c = 1$ matrix quantum mechanics. This is established at the level of spectrum of states, since the partition function without perturbations is known to be T-dual [29]. Also, a formal argument has been given [16] that the winding correlators, like the momentum correlators, are given by a Toda hierarchy.[2] However, to our knowledge, beyond this result and a computation in [19], there has been no direct comparison of correlators in the momentum and winding sectors.[3] A convincing test of T-duality would consist of computing pure-momentum correlators in terms of free fermion eigenvalues, T-dualising the answers and comparing them with pure-winding correlators computed from the nonsinglet Hamiltonian. Ideally this should even be done beyond tree level. Although we will not be able to carry out such a test here, we have tried to systematise one side of the duality in a way that can be eventually compared with the other side when nonsinglet computations become more practicable.

In particular, the most direct way to check T-duality comes from comparing two-point functions. Accordingly we work out all two-point functions of momentum modes. In ref. [19], the two-point function of unit-momentum modes was computed and an attempt made to match the leading result with a computation in the first nonsinglet sector of the matrix model, namely the adjoint sector. The comparison revealed the presence of unexplained normalisation factors. It was pointed out in ref. [19] that if one could compute two-point functions of more general winding modes, namely $\langle \mathcal{T}_{-nR} \mathcal{T}_{nR} \rangle$, one might be able to shed some light on these normalisation factors. With this motivation we have performed this computation and obtained a simple explicit result, again as a function of μ and R and for all n. In a later section we discuss the relation to the non-singlet sectors.

Our initial computations have been performed using both the MQM and a model of constant matrices called the Normal Matrix Model (NMM) [21], with perfect agreement between the answers. In the former case we used the known infinite-radius correlators in the physical MQM (real and noncompact time) [31], and a formula which converts these to the correlators for Euclidean compact time [32]. In the latter case, we will describe how

[2] For a discussion of T-duality in type 0A,B matrix models, see ref. [30].

[3] In addition to pure momentum or pure winding correlators, one would also like to know the correlators for a mixture of momentum and winding modes. In this case one has no choice but to tackle the difficult nonsinglet sector problem. The system is not expected to be integrable and the correlation functions are not known so far.

one performs computations after reducing the NMM to eigenvalues. One surprise emerging from comparison of the two approaches is that the NMM successfully computes correlators even when the matrices are of finite rank N, a stronger property than was claimed in ref. [21], who did however suggest that the model contains some information even at finite N. We find that it actually contains *complete* information at finite N in the following sense: given a correlator, there is a minimum value $N_{\rm min}$ such that this correlator when computed in the NMM gives the correct result, to all orders in $1/\mu^2$, for all $N > N_{\rm min}$. This makes the NMM a potentially powerful combinatoric tool. We then go on to demonstrate its power by deriving a combinatoric formula for the general correlation function $\langle (T_{-1/R})^n (T_{1/R})^n \rangle$ for any n.

We start in section 2 by describing the two relevant matrix models, Matrix Quantum Mechanics and Normal Matrix Model. The former is too well-known to need a detailed discussion and we skip directly to the computational techniques and answers. For the latter, we review the model in some detail, with special attention to the role of the matrix rank N. In section 3 we work out some relevant correlators as a function of μ and R from MQM. In section 4 we reproduce these correlators from the NMM, where we note the phenomenon that for a fixed correlator, the NMM at any N greater than a minimum value gives the complete answer. After a discussion of why this works, we use this property to derive a combinatoric formula for correlators of any number of unit momentum modes. In section 5 we discuss applications of these results to some physically interesting problems, and conclude in section 6. Several computational details are presented in the appendices.

2. Matrix quantum mechanics and normal matrix model

2.1 Matrix quantum mechanics

Matrix Quantum Mechanics is a model of a single $N \times N$ hermitian time-dependent matrix $M(t)$. In the absence of perturbations, the partition function of the model is given by:

$$\mathcal{Z}^{(N)}_{\rm MQM} = \int [dM] \exp\left[-N \int dt \, {\rm tr}\left((D_t M)^2 + M^2 \right) \right] \tag{2.1}$$

where $D_t M \equiv \dot{M} + i[A_t, M]$ is the covariant derivative with respect to the time component of a gauge field.

The gauge field acts as a Lagrange multiplier and projects the model to the singlet sector, which is a system of N non-interacting non-relativistic fermions moving in an inverted harmonic oscillator potential. In the double-scaling limit, the Fermi sea is filled nearly to the top and the number of fermions is taken to infinity. The scaled distance to the top of the potential, μ, is kept finite and corresponds to the cosmological constant. This model provides a description of 2D string theory, with μ^{-1} playing the role of the string coupling g_s.

The physical modes of 2D string theory can be constructed in terms of fermion eigenvalues. In [31] this model was used to calculate correlation functions of $c = 1$ string theory at infinite radius. One starts by computing correlators of free-fermion bilinears, which in turn can be used to extract correlators of the loop operators:

$$\mathcal{O}(k,\ell) = \int dt\, e^{ikt}\, \mathrm{tr}\, e^{-\ell M(t)} \tag{2.2}$$

Extracting the leading behaviour of these loops for small ℓ, one has

$$\mathcal{O}(k,\ell) \sim \ell^{|k|}\, T_k \tag{2.3}$$

The T_k are identified with the $c = 1$ string theory tachyons. When compared with the corresponding operators in Liouville theory, there is a change of normalisation:

$$T_k|_{\rm MQM} = \Gamma(|k|)\ T_k|_{\rm Liouville} \tag{2.4}$$

However this fact will not be relevant for us, since in what follows we will always work with the operators T_k in the MQM basis, i.e. the l.h.s. of the above equation.

When the time direction is Euclidean and compact, we are in the finite temperature theory. Starting from the infinite-radius correlator, one can show [32] that correlators in the Euclidean theory at finite radius are obtained as:

$$\langle T_{q_1} T_{q_2} \cdots T_{q_n}\rangle_R = \frac{\frac{1}{2R}\partial_\mu}{\sin\left(\frac{1}{2R}\partial_\mu\right)} \langle T_{q_1} T_{q_2} \cdots T_{q_n}\rangle_\infty \tag{2.5}$$

In addition one must replace the momentum-conserving δ-function as:

$$\delta\Big(\sum_i q_i\Big) \to R\, \delta_{\sum_i q_i, 0} \tag{2.6}$$

The above prescriptions follow from the fact that the compact radial direction introduces an additional factor in the loop momentum integrals of the infinite-radius calculation, and this factor can now be taken out of the integrals whence it becomes a differential operator acting on the infinite-radius answer.[4]

In the finite-temperature theory, the above modes can be thought of as carrying "momentum" in the time direction. In this situation one also expects to find winding modes corresponding to the thermal scalars of finite-temperature string theory. Many physical properties of string theory are encoded in these degrees of freedom, which are therefore quite important to study. To find them in the matrix model we must go beyond the singlet sector, in which the gauge field is topologically trivial and can be gauged away. Consider the gauge-invariant Wilson-Polyakov loop variable:

$$W_{\mathcal{R}} = \mathrm{tr}_{\mathcal{R}}\, P\, \exp(i \oint A_t dt) \tag{2.7}$$

where the trace is performed in the representation $\mathcal{R}$ of SU(N). When $\mathcal{R}$ is the fundamental representation, this is to be associated with a unit winding mode:

$$\mathcal{W}_{\mathcal{R}=N} \sim \mathcal{T}_R \tag{2.8}$$

[4]It is also possible to calculate correlators directly at finite radius using the "reflection coefficient" formalism of ref. [20]. Though we will not use this here, it would be interesting to know if our explicit results follow as easily in that approach.

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Similarly the trace in the anti-fundamental will be $\mathcal{T}_{-R}$. One can also have loops where the trace continues to be in the fundamental but the contour winds multiple times over the Euclidean time direction. Computation of the correlation functions of all these Wilson-Polyakov loops is done by observing that in their presence, the matrix model receives contributions from definite non-singlet sectors. In these sectors it reduces to eigenvalue fermions but now with mutual interactions. For example, the two-point function of unit winding modes can be identified as follows:

$$\langle \mathcal{T}_{-R}\mathcal{T}_R\rangle\Big|_{\textbf{Liouville theory}} \sim \langle \mathcal{W}_{\bar{N}}\mathcal{W}_N\rangle\Big|_{\textbf{MQM}} \sim \langle \mathcal{W}_{\textbf{adjoint}}\rangle\Big|_{\textbf{MQM}} \tag{2.9}$$

Thus computing the partition function of MQM in the adjoint sector determines the two-point function of winding modes. Since in principle this is an independent computation from that of the momentum tachyon correlators, it can actually be used to check T-duality of the $c = 1$ string. We will return to this issue in a subsequent section.

2.2 Normal Matrix Model

The Normal Matrix Model (NMM) [21] is a relatively simple model of a complex matrix Z and its Hermitian adjoint, with the constraint that the two commute (hence Z is said to be "normal"). The potential is polynomial with an additional logarithmic piece. The matrix Z is constant rather than time-dependent, so in this sense it is more similar to the $c < 1$ string backgrounds which do not have a time direction.[5]

The NMM is proposed to describe the correlators of the $c = 1$ string to all orders in perturbation theory, as follows. Let us introduce its partition function:

$$\mathcal{Z}^{(N)}_{\rm NMM}(\nu, t, \bar{t}) = \int [dZdZ^\dagger]\; e^{\mathbf{tr}\left(-\nu(ZZ^\dagger)^R + \left(R\nu - N + \frac{R-1}{2}\right)\log ZZ^\dagger - \nu\sum_{k=1}^{\infty}(t_k Z^k + \bar{t}_k {Z^\dagger}^k)\right)} \tag{2.10}$$

Here R, ν are some (in general, complex) parameters, which will correspond to the compactification radius of Euclidean time and the cosmological constant respectively. The parameters t_k, $\bar{t}_k$ are couplings to the gauge-invariant operators ${\rm tr}Z^k$, ${\rm tr}{Z^\dagger}^k$ and Z, $Z^\dagger$ are $N \times N$ matrices satisfying:

$$[Z, Z^\dagger] = 0 \tag{2.11}$$

The operators ${\rm tr}Z^k, {\rm tr}{Z^\dagger}^k$ are identified with the tachyons $T_{k/R}, T_{-k/R}$ of momentum $\pm\frac{k}{R}$ respectively.

Since the matrix Z commutes with its adjoint, the two can be simultaneously diagonalised. The diagonalising matrices drop out of the action leaving behind Vandermonde factors. It turns out that one gets a single power of the Vandermonde for the eigenvalues $z_1, z_2, \ldots, z_N$ of Z, together with its complex conjugate corresponding to $Z^\dagger$. Thus, for example, the partition function at $t_k = \bar{t}_k = 0$ is:

$$\mathcal{Z}_{\rm NMM} = \int \prod_{i=1}^{N} d^2 z_i \prod_{i<j} |z_i - z_j|^2\; e^{-\nu\sum_{i=1}^{N}(z_i\bar{z}_i)^R + \left(R\nu - N + \frac{R-1}{2}\right)\sum_{i=1}^{N}\log z_i\bar{z}_i} \tag{2.12}$$

with an obvious generalisation to include the tachyon perturbations.

[5] Perhaps this is the underlying reason why the NMM describes Euclidean $c = 1$ strings at an arbitrary radius R, but does not have a simple $R \to \infty$ limit where one might recover the Lorentzian theory.

At $t_k = \bar{t}_k = 0$, it can be shown (though not directly from the action) that the NMM is invariant under the T-duality operation:

$$R \to \frac{1}{R}, \qquad \mu \to \mu\, R \tag{2.13}$$

This invariance is broken by the presence of momentum modes. Indeed, after T-duality, the tachyons $T_{\pm k/R}$ of the $c = 1$ string turn into winding modes of $\pm k$ units of winding, or equivalently (in vertex-operator language) of left/right momentum $(p_L, p_R) = \pm(kR, -kR)$. In what follows, these modes will be denoted $\mathcal{T}_{kR}, \mathcal{T}_{-kR}$.

In [21] two distinct equivalences between the NMM and the $c = 1$ string were proposed. The first, referred to as "Model I", requires us to take the large-N limit of the NMM. The result in this case was that:

$$\mathcal{Z}_{c=1}(\mu, t, \bar{t}) = \lim_{N\to\infty} \mathcal{Z}^{(N)}_{\rm NMM}(\nu, t, \bar{t}), \tag{2.14}$$

after the analytic continuation $\nu = -i\mu$.

However, another equivalence, "Model II", was proposed which did not involve a large-N limit. It was argued that the $c = 1$ string theory can be obtained from the NMM at *finite* N, provided ν is set to the special value $\frac{N}{R}$ (note that this corresponds to an imaginary cosmological constant):

$$\mathcal{Z}_{c=1}\left(\mu = i\tfrac{N}{R}, t, \bar{t}\right) = \mathcal{Z}^{(N)}_{\rm NMM}\left(\nu = \tfrac{N}{R}, t, \bar{t}\right), \tag{2.15}$$

In other words, the claim[6] is that an NMM calculation for a fixed integer value of N determines $\mathcal{Z}_{c=1}$ for a particular (imaginary) value of μ, namely

$$\mu = i\,\frac{N}{R} \tag{2.16}$$

If we T-dualise the above considerations so that $t, \bar{t}$ become couplings to winding tachyons, this relation becomes

$$\mu = iN \tag{2.17}$$

The above results seem to indicate that for finite N we can only generate the answer at a fixed μ, in which case we would never obtain the perturbative expansion in powers of $1/\mu^2$. However, below we will compute winding correlators using the NMM, and will see that it turns out much more powerful than expected. It actually does reproduce the entire perturbative correlators, as functions of μ and R, even at finite values of N. Evidence for this fact, as well as an explanation of it, will be provided in subsequent sections.

[6]The authors of ref. [21] stated this a little differently: that one obtains $c = 1$ string amplitudes as a function of μ by computing NMM correlators as a function of N and μ, and then continuing N to the imaginary value $-i\mu R$. This procedure is less well-defined, as it requires us to make a discrete parameter continuous.

3. Correlators from matrix quantum mechanics

3.1 Two-point functions

We start by presenting formulae for the two-point function $\langle T_{-n/R} T_{n/R} \rangle$ to all orders in $\frac{1}{\mu^2}$, from the Matrix Quantum Mechanics (MQM) approach. We will derive these formulae, valid at arbitrary radius, starting from the infinite-radius formulae presented in [31]. We start by quoting the closed-form expression for the infinite-radius two-point function $\langle T_{-q} T_q \rangle$, or more precisely the first derivative of the two-point function with respect to the cosmological constant, which is actually more convenient for our purposes:

$$\partial_\mu \langle T_{-q} T_q \rangle_\infty = (\Gamma(-q))^2 \ \text{Im}\, e^{i\pi q/2} \left(\frac{\Gamma\left(\frac{1}{2} - i\mu + q\right)}{\Gamma\left(\frac{1}{2} - i\mu\right)} - \frac{\Gamma\left(\frac{1}{2} - i\mu\right)}{\Gamma\left(\frac{1}{2} - i\mu - q\right)} \right), \tag{3.1}$$

where $q > 0$. For clarity of presentation we will drop the leg-pole factors $(\Gamma(-q))^2$ in what follows, keeping in mind that they can be restored whenever needed.

Now we obtain the corresponding amplitudes at a finite radius R, using eqs. (2.5) and (2.6):

$$\langle T_{-q} T_q \rangle_R = R \, \frac{\frac{1}{2R}}{\sin\left(\frac{1}{2R}\partial_\mu\right)} \ \text{Im}\, e^{i\pi q/2} \left(\frac{\Gamma\left(\frac{1}{2} - i\mu + q\right)}{\Gamma\left(\frac{1}{2} - i\mu\right)} - \frac{\Gamma\left(\frac{1}{2} - i\mu\right)}{\Gamma\left(\frac{1}{2} - i\mu - q\right)} \right)$$

where the first factor of R comes from the replacement of the δ-function by a Kronecker δ as in eq. (2.6). The differential operator in front is real and acts only on functions of μ, so it can be moved inside and we thus need to evaluate

$$\frac{1}{2\sin\left(\frac{1}{2R}\partial_\mu\right)} \left(\frac{\Gamma\left(\frac{1}{2} - i\mu + q\right)}{\Gamma\left(\frac{1}{2} - i\mu\right)} - \frac{\Gamma\left(\frac{1}{2} - i\mu\right)}{\Gamma\left(\frac{1}{2} - i\mu - q\right)} \right)$$

This can be done very easily by expanding the operator as follows

$$\frac{1}{2\sin\left(\frac{1}{2R}\partial_\mu\right)} = -i \sum_{j=0}^{\infty} e^{i\left(j+\frac{1}{2}\right)\frac{1}{R}\partial_\mu}$$

Using this we get the required expression as

$$-i \sum_{j=0}^{\infty} \left(\frac{\Gamma\left(\frac{1}{2} - i\mu + q + \frac{j}{R} + \frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu + \frac{j}{R} + \frac{1}{2R}\right)} - \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{j}{R} + \frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - q + \frac{j}{R} + \frac{1}{2R}\right)} \right) \tag{3.2}$$

Next, we choose $q = n/R$. We see that the j^{th} term from the first sum cancels the $(j+n)^{\text{th}}$ term from the second sum. So only the $j = 0, 1, \ldots, n-1$ terms from the second sum remain. Defining $r = n - j$, the above expression becomes:[7]

$$\langle T_{-n/R} T_{n/R} \rangle = \text{Re}\, e^{i\pi n/2R} \sum_{r=1}^{n} \frac{\Gamma\left(\frac{1}{2} - i\mu + \left(r - \frac{1}{2}\right)\frac{1}{R}\right)}{\Gamma\left(\frac{1}{2} - i\mu + \left(r - n - \frac{1}{2}\right)\frac{1}{R}\right)} \tag{3.3}$$

[7] Here and in what follows, we drop the R subscript in the correlators wherever it is obvious that they are at finite R.

In order to obtain the expansion of this expression in powers of $1/\mu^2$, we can rewrite it in terms of the special functions:

$$\mathcal{F}^{\pm}(a,b;\mu) \equiv \frac{\Gamma(\frac{1}{2}-i\mu+a)}{\Gamma(\frac{1}{2}-i\mu+b)} \pm \frac{\Gamma(\frac{1}{2}-i\mu-b)}{\Gamma(\frac{1}{2}-i\mu-a)} \tag{3.4}$$

defined in eq. (B.2) of ref. [31]. We have:

$$\langle T_{-n/R} T_{n/R}\rangle = \mathrm{Re}\; e^{i\pi n/2R} \sum_{r=1}^{n/2} \mathcal{F}^{+}\Big((r-\tfrac{1}{2})\tfrac{1}{R}, (r-n-\tfrac{1}{2})\tfrac{1}{R};\mu\Big), \qquad n \text{ even} \tag{3.5}$$

$$= \mathrm{Re}\; e^{i\pi n/2R}\left(\tfrac{1}{2}\mathcal{F}^{+}\left(\tfrac{n}{2R}, -\tfrac{n}{2R};\mu\right) + \sum_{r=1}^{(n-1)/2} \mathcal{F}^{+}\Big((r-\tfrac{1}{2})\tfrac{1}{R}, (r-n-\tfrac{1}{2})\tfrac{1}{R};\mu\Big)\right), \; n \text{ odd}$$

Next we use the asymptotics for large μ:

$$\mathcal{F}^{+}(a,b;\mu) = e^{-i\pi(a-b)/2}\, \mu^{a-b}\, f(a,b;\mu) \tag{3.6}$$

where $f(a,b;\mu)$ is a power series in $\frac{1}{\mu^2}$ with real coefficients and starting with a constant term:

$$f(a,b;\mu) = 2 - \tfrac{1}{12}(a-b)(a-b-1)\left(3(a+b)^2 - (a-b) - 1\right)\frac{1}{\mu^2} + \mathcal{O}\left(\frac{1}{\mu^4}\right) \tag{3.7}$$

It follows that, for even n:

$$\begin{aligned} \langle T_{-n/R} T_{n/R}\rangle &= \mathrm{Re}\; \mu^{n/R} \sum_{r=1}^{n/2} f\Big((r-\tfrac{1}{2})\tfrac{1}{R}, (r-n-\tfrac{1}{2})\tfrac{1}{R};\mu\Big) \\ &= \mu^{n/R} \sum_{r=1}^{n/2} f\Big((r-\tfrac{1}{2})\tfrac{1}{R}, (r-n-\tfrac{1}{2})\tfrac{1}{R};\mu\Big) \\ &= \left| \sum_{r=1}^{n} \frac{\Gamma(\frac{1}{2}-i\mu+(r-\frac{1}{2})\frac{1}{R})}{\Gamma(\frac{1}{2}-i\mu+(r-n-\frac{1}{2})\frac{1}{R})} \right| \end{aligned} \tag{3.8}$$

The first step above follows because the function f is real. The final equality is true for all n, and not just even values. This then is the complete answer for the perturbative expansion of two-point functions of momentum correlators at arbitrary radius.

Specialising to $n = 1$, we find the following expression, which will be useful later on:

$$\langle T_{-1/R} T_{1/R}\rangle = \left| \frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2}-i\mu-\frac{1}{2R}\right)} \right| \tag{3.9}$$

After a T-duality

$$R \to 1/R, \quad \mu \to \mu R \tag{3.10}$$

we get the unit-winding two-point function

$$\langle \mathcal{T}_{-R} \mathcal{T}_{R}\rangle = \left| \frac{\Gamma\left(\frac{1}{2}-i\mu R+\frac{R}{2}\right)}{\Gamma\left(\frac{1}{2}-i\mu R-\frac{R}{2}\right)} \right| \tag{3.11}$$

This expression was recently derived by Maldacena [19]. We should note that the above answer has to be multiplied by the leg pole factor $\big(\Gamma(-R)\big)^2$, which we dropped after eq. (3.1).

3.2 Four-point functions

In this section we turn to the computation of higher point functions. In particular, we extend the results for the four-point function from MQM to finite R and then specialise to the case of unit winding modes. In this case we will be able to find an explicit all-orders result after summing an infinite series.

Upto leg pole factors (which can be unambiguously restored when needed) the connected four-point function at infinite radius is [31]:

$$\partial_\mu \langle (T_{-q}T_q)^2\rangle_\infty^{\rm conn} = {\rm Im}\; e^{i\pi q}\Bigg[\mathcal{F}^+(2q,0;\mu) - \mathcal{F}^+(q,-q;\mu) + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}2\left(\frac{\Gamma(-q+n)}{\Gamma(-q)}\right)^2 \times\left(\frac{\Gamma\left(2q-n+\frac{1}{2}-i\mu\right)}{\Gamma\left(\frac{1}{2}-i\mu\right)} - \frac{\Gamma\left(q-n+\frac{1}{2}-i\mu\right)}{\Gamma\left(-q+\frac{1}{2}-i\mu\right)}\right)\Bigg], \tag{3.12}$$

where $q>0$ and the function $\mathcal{F}^+$ is defined in eq. (3.4).

Substituting eq. (3.4) in eq. (3.12) we have

$$\partial_\mu \langle (T_{-q}T_q)^2\rangle_\infty^{\rm conn} = {\rm Im}\; e^{i\pi q}\Bigg[\frac{\Gamma\left(\frac{1}{2}-i\mu+2q\right)}{\Gamma\left(\frac{1}{2}-i\mu\right)} + \frac{\Gamma\left(\frac{1}{2}-i\mu\right)}{\Gamma\left(\frac{1}{2}-i\mu-2q\right)} - 2\frac{\Gamma\left(\frac{1}{2}-i\mu+q\right)}{\Gamma\left(\frac{1}{2}-i\mu-q\right)} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\Gamma(-q+n)}{\Gamma(-q)}\right)^2\left(\frac{\Gamma\left(\frac{1}{2}-i\mu+2q-n\right)}{\Gamma\left(\frac{1}{2}-i\mu\right)} - \frac{\Gamma\left(\frac{1}{2}-i\mu+q-n\right)}{\Gamma\left(\frac{1}{2}-i\mu-q\right)}\right)\Bigg] \tag{3.13}$$

The connected finite-R amplitude is, therefore

$$\langle (T_{-q}T_q)^2\rangle_R^{\rm conn} = R\,\frac{\frac{1}{2R}\partial_\mu}{\sin\left(\frac{1}{2R}\partial_\mu\right)}\langle (T_{-q}T_q)^2\rangle_\infty^{\rm conn}$$

We use the expansion eq. (3.2) of the differential operator and set $q=1/R$ to get

$$\langle (T_{-1/R}T_{1/R})^2\rangle^{\rm conn} = {\rm Re}\; e^{i\pi/R}\Bigg(-\frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{3}{2R}\right)}{\Gamma\left(\frac{1}{2}-i\mu-\frac{1}{2R}\right)} + \frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2}-i\mu-\frac{3}{2R}\right)} - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\Gamma(-\frac{1}{R}+n)}{\Gamma(-\frac{1}{R})}\right)^2\frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{3}{2R}-n\right)}{\Gamma\left(\frac{1}{2}-i\mu-\frac{1}{2R}\right)}\Bigg) \tag{3.14}$$

It is convenient to add and subtract a term corresponding to $n=0$ in the summation. This extends the sum from 0 to ∞, while the subtracted term changes the sign of the first term above, after which the first two terms combine into an $\mathcal{F}^+$. Thus we get:

$$\langle (T_{-1/R}T_{1/R})^2\rangle^{\rm conn} = {\rm Re}\; e^{i\pi/R}\Bigg(\mathcal{F}^+(\tfrac{3}{2R},-\tfrac{1}{2R};\mu) - 2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\Gamma(-\frac{1}{R}+n)}{\Gamma(-\frac{1}{R})}\right)^2\frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{3}{2R}-n\right)}{\Gamma\left(\frac{1}{2}-i\mu-\frac{1}{2R}\right)}\Bigg) \tag{3.15}$$

The sum is now easy to evaluate using the integral representations for the three Γ-functions in the numerator that depend on n (see appendix B). This finally leads to:

$$\begin{aligned}\langle (T_{-1/R}T_{1/R})^2\rangle^{\rm conn} &= {\rm Re}\ e^{i\pi/R}\left(\mathcal{F}^+(\tfrac{3}{2R},-\tfrac{1}{2R};\mu) - 2\Big(\tfrac12\mathcal{F}^+(\tfrac{1}{2R},-\tfrac{1}{2R};\mu)\Big)^2\right)\\ &= \left|\mathcal{F}^+(\tfrac{3}{2R},-\tfrac{1}{2R};\mu) - 2\Big(\tfrac12\mathcal{F}^+(\tfrac{1}{2R},-\tfrac{1}{2R};\mu)\Big)^2\right| \end{aligned} \tag{3.16}$$

One can verify that the two terms above are, respectively, the full (connected plus disconnected) correlator, and its disconnected part.

4. Correlators in the finite-N normal matrix model

Having obtained explicit expressions for all two-point and a particular four-point function from the MQM, as a function of the cosmological constant μ and radius R, we now attempt to recover the same results from the NMM. This first of all provides a test of the NMM and its effectiveness. But once we explore the systematics it will become clear that we can compute much more. In fact, we will obtain a complete combinatorial formula for the $2n$-point functions of unit-momentum correlators. Via T-duality, this determines the corresponding winding correlators. We expect this to be useful in determining the full vortex condensate to all orders in perturbation theory.

As mentioned before, in the process of studying the NMM we will encounter a rather surprising result: for the purpose of computing correlators, one can actually take N to be a small finite value and yet obtain the correct answer *as a function of* μ. The finite value of N will be determined by the operators whose correlators we are calculating. For this purpose it is convenient to classify tachyon correlators into sectors labelled by an integer, the total positive momentum P flowing through that correlator, measured in units of $1/R$. For example in $\langle T_{-k_1/R}T_{-k_2/R}T_{m_1/R}T_{m_2/R}\rangle$, where k_1, k_2, m_1, m_2 are all positive, the total positive momentum is $P = m_1 + m_2 = k_1 + k_2$. This number will determine the minimum value of N required in the NMM to compute these correlators. In what follows we will first consider all correlators in the sectors $P = 1$ and $P = 2$. In the former case there is only a single two-point function, while in the latter case we have two, three and four-point functions. After presenting some examples we will discuss why the theory works in this way.

4.1 Two-point functions: examples

Example: $n = 1$

We begin by computing the two point function of the unit momentum operator. Since total momentum is conserved, this operator is paired with the one of negative unit momentum. So we will calculate the two point function $\langle T_{-1/R}T_{1/R}\rangle$ of unit momentum operators.

We first calculate the partition function of NMM at $N = 1$:

$$\mathcal{Z}^{N=1}_{\rm NMM}(t=0) = \int dz d\bar z\, e^{-\nu(z\bar z)^R + \left(R\nu - 1 + \frac{R-1}{2}\right)\log z\bar z} \tag{4.1}$$

Setting $z = \sqrt{m}\, e^{i\theta}$, $dzd\bar{z} \to dm\, d\theta$, we have:

$$\begin{aligned} \mathcal{Z}_{\rm NMM}^{N=1}(t=0) &= \int_0^\infty \int_0^{2\pi} dm\, d\theta\, e^{-\nu m^R + \left(R\nu - 1 + \frac{R-1}{2}\right)\log m} \\ &= 2\pi \int_0^\infty dm\, m^{\left(R\nu - 1 + \frac{R-1}{2}\right)} e^{-\nu m^R} \\ &= \frac{2\pi}{R} \nu^{-\left(\nu + \frac{1}{2} - \frac{1}{2R}\right)} \Gamma\left(\nu + \tfrac{1}{2} - \tfrac{1}{2R}\right) \end{aligned} \tag{4.2}$$

As a function of ν, this is not the correct partition function of the $c = 1$ string, but it reduces to the correct partition function if in the above expression we set $\nu = \frac{1}{R}$ and compare this with $\mathcal{Z}_{c=1}(\frac{i}{R}, t = 0, \bar{t} = 0)$. This fact is a direct consequence of the claim in ref. [21], see eq. (2.15). It is also worth noting that the partition function at $N = 1$ is not invariant under T-duality. In fact, T-duality in the NMM partition function is recovered only in the limit $N \to \infty$. This makes it clear that the correct partition function, as a function of μ and R, can never be recovered at finite N.

For correlators, things are quite different, as we will now see. For the two-point function, we find:

$$\begin{aligned} \partial_{-1}\partial_1 \mathcal{Z}_{\rm NMM}^{N=1}(t=0) &= \int_0^\infty \int_0^{2\pi} dm\, d\theta\, m\, e^{-\nu m^R + \left(R\nu - 1 + \frac{R-1}{2}\right)\log m} \\ &= \frac{2\pi}{R} \nu^{-\left(\nu + \frac{1}{2} + \frac{1}{2R}\right)} \Gamma\left(\nu + \tfrac{1}{2} + \tfrac{1}{2R}\right) \end{aligned} \tag{4.3}$$

From eq. (4.2) and eq. (4.3) we have:

$$\partial_{-1}\partial_1 \ln \mathcal{Z}_{\rm NMM}^{N=1}(t=0) = \nu^{-\frac{1}{R}} \frac{\Gamma\left(\nu + \frac{1}{2} + \frac{1}{2R}\right)}{\Gamma\left(\nu + \frac{1}{2} - \frac{1}{2R}\right)} \tag{4.4}$$

Finally, we have to analytically continue $\nu = -i\mu$. The result is complex, but can easily be seen to have the form of an overall phase times a real power series in $1/\mu^2$. Dropping the phase is then equivalent to taking the modulus of the above expression. This gives:

$$\langle T_{-1/R} T_{1/R} \rangle_{\rm NMM}^{N=1} = \mu^{-\frac{1}{R}} \left| \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{1}{2R}\right)} \right| \tag{4.5}$$

which agrees with eq. (3.9) upto the prefactor, $\mu^{-1/R}$, which indicates that the "tachyons" of the NMM are normalised differently from those of MQM. Indeed we will argue later that the relationship is:

$$T_{n/R}|_{\rm NMM} = \mu^{-n/2R}\, T_{n/R}|_{\rm MQM} \tag{4.6}$$

We have discovered the surprising result that the exact two-point correlator of unit momentum tachyons is correctly calculated (as a function of μ and R) using only the 1×1 Normal Matrix Model! According to eq. (2.15), we should have expected the result to be correct only for $\mu = i/R$. We will see that a similar feature holds for all two-point correlators, though the minimum required value of N depends on the correlator under consideration. Later we will extend this observation to higher-point correlators.

Example: $n = 2$

We consider another example, the correlator $\langle T_{-2/R} T_{2/R}\rangle$. In this case, according to the prediction in eq. (2.15), we can perform a calculation at $N = 1$ and the result so obtained will be valid at the special value of the cosmological constant $\nu = 1$. However, we now face a puzzle. In the NMM at $N = 1$, one cannot distinguish the four correlators:

$$\langle T_{-2/R} T_{2/R}\rangle, \quad \langle T_{-2/R} T_{1/R} T_{1/R}\rangle, \quad \langle T_{-1/R} T_{-1/R} T_{2/R}\rangle, \quad \langle T_{-1/R} T_{-1/R} T_{1/R} T_{1/R}\rangle \tag{4.7}$$

because all of these are represented by the same NMM correlator $\langle z^2 \bar{z}^2\rangle$. Therefore, assuming eq. (2.15) continues to hold, either it has to be the case that all four correlators become the same at $\nu = \frac{1}{R}$, or else at best we can only hope to obtain some linear combination of them.

The calculation is straightforward and upon continuing to $\nu = -i\mu$ and taking the modulus, we find:

$$\langle T_{-2/R} T_{2/R}\rangle^{N=1}_{\rm NMM} = \mu^{-2/R} \left| \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{3}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{1}{2R}\right)} \right| \tag{4.8}$$

This can be compared with the known result from eq. (3.8). Specialising to the present case, and changing to the NMM normalisation via eq. (4.6) gives us:

$$\langle T_{-2/R} T_{2/R}\rangle = \left| \frac{\Gamma(\frac{1}{2} - i\mu + \frac{1}{2R})}{\Gamma(\frac{1}{2} - i\mu - \frac{3}{2R})} + \frac{\Gamma(\frac{1}{2} - i\mu + \frac{3}{2R})}{\Gamma(\frac{1}{2} - i\mu - \frac{1}{2R})} \right| \tag{4.9}$$

Comparing eqs. (4.8), (4.9), we see that the NMM result for this correlator at $N = 1$ is not correct. This is not a surprise. But now we see that it is incorrect even at the special value $\mu = i/R$, which appears to contradict eq. (2.15). As we will see, this is due to the fact that the same NMM correlator can describe different tachyon correlation functions for low N. Indeed, one can check that the answer we have obtained at $N = 1$ in eq. (4.8) is actually a linear combination of the correlators in eq. (4.7) as calculated from matrix quantum mechanics.

Let us continue by evaluating the NMM correlator at $N = 2$. In this case the operator we are dealing with is $T_{2/R} \sim {\rm tr}Z^2$ which is linearly independent of $(T_{1/R})^2 \sim ({\rm tr}Z)^2$ once Z is a 2×2 matrix, so there is no longer a risk of mixing for the operators in eq. (4.7). The computation is given in an appendix, and leads to the answer eq. (A.5), which after changing to the NMM normalisation is:

$$\langle T_{-2/R} T_{2/R}\rangle^{N=2}_{\rm NMM} = \left| \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{3}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{1}{2R}\right)} + \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{3}{2R}\right)} \right| \tag{4.10}$$

Following eq. (2.15) we would expect that this should give the correct answer for $\mu = 2i/R$. But now there is a surprise, since in fact it agrees perfectly with the MQM result eq. (4.9) for *all* values of μ. Thus for the purposes of calculating $\langle T_{-2/R} T_{2/R}\rangle$ in $c = 1$ string theory, to all orders in the string coupling, a 2×2 matrix model is sufficient.

To summarise, we have found evidence that an NMM calculation of tachyon correlators at finite N (where the minimum required value of N depends on the correlator in question) gives the correct tachyon correlators for the $c = 1$ string, to all orders in perturbation theory. Below we will collect more evidence for this property, which appears to go far beyond the result of ref. [21] as stated in eq. (2.15) above.

4.2 Two-point functions: general case

Let us now consider the general case $\langle T_{-n/R} T_{n/R} \rangle$. and try to derive this result from the NMM. We will find that for this correlator, the NMM with $N = n$ is sufficient to give the correct result. Indeed, when we compute in the $N \times N$ NMM starting at $N = 1$ and increasing N in integer steps, we obtain the right $c = 1$ string correlator (as a function of μ) as long as $N \geq n$, though not for $N < n$. Thus the NMM calculation "stabilises" at a certain minimum value of N.

Since we will be computing normalised correlators, we start by computing the (unperturbed) partition function at a general value of N. This is given by

$$\mathcal{Z}^N_{\rm NMM}(t=0) = \int_0^\infty \prod_{r=1}^N dm_r \int_0^{2\pi} \prod_{r=1}^N d\theta_r \tag{4.11}$$
$$\times \prod_{j<k}^N \Big(m_j + m_k - \sqrt{m_j m_k}(e^{i\theta_{jk}} + e^{-i\theta_{jk}}) \Big)\, e^{-\nu\left(\sum_{r=1}^N m_r^R\right) + \left(R\nu - N + \frac{R-1}{2}\right)\left(\sum_{r=1}^N \log m_r\right)}$$

The next step is to perform the integration over the θ's. In general this will be quite tedious, because one has to pick out terms which are independent of θ by expanding out the Vandermonde factor. However, we notice that since the above expression is invariant under permutations of m's, we can determine all terms surviving the θ integrals if we know just one of them, by permuting the m's among themselves.

The first such term is just the product of the first term from each of the Vandermonde factors, which is $m_1^{N-1} m_2^{N-2} \cdots m_{N-1}$. Thus we have, after evaluating the θ integrals

$$\mathcal{Z}^N_{\rm NMM}(t=0) = (2\pi)^N N! \int_0^\infty \prod_{r=1}^N dm_r \prod_{j=1}^{N-1} m_j^{N-j} \tag{4.12}$$
$$\times\, e^{-\nu\left(\sum_{r=1}^N m_r^R\right) + \left(R\nu - N + \frac{R-1}{2}\right)\left(\sum_{r=1}^N \log m_r\right)}$$
$$= (2\pi)^N N! \prod_{r=1}^N \nu^{-\left(\nu + \frac{1}{2} - \left(r - \frac{1}{2}\right)\frac{1}{R}\right)} \Gamma\Big(\nu + \tfrac{1}{2} - \left(r - \tfrac{1}{2}\right)\tfrac{1}{R}\Big)$$

¿From now on we will restrict to the case $N = n$.

The next step is to compute the two point function and then normalise by the above partition function. We have:

$$\partial_{-n}\partial_n \mathcal{Z}^{N=n}_{\rm NMM}(t=0) = \int \prod_{r=1}^n d^2 z_r \prod_{j<k}^n |z_j - z_k|^2 \left(\sum_{l=1}^n z_l^n\right)\left(\sum_{l=1}^n \bar{z}_l^n\right)$$
$$\times e^{-\nu\left(\sum_{r=1}^n (z_r \bar{z}_r)^R\right) + \left(R\nu - n + \frac{R-1}{2}\right)\left(\sum_{r=1}^n \log z_r \bar{z}_r\right)}$$
$$= \int_0^\infty \prod_{r=1}^n dm_r \int_0^{2\pi} \prod_{r=1}^n d\theta_r \prod_{j<k}^n \Big(m_j + m_k - \sqrt{m_j m_k}(e^{i\theta_{jk}} + e^{-i\theta_{jk}}) \Big)$$

$$\times \left(\sum_{r=1}^{n} \left(\sqrt{m_r}\right)^n e^{in\theta_r} \right) \left(\sum_{r=1}^{n} \left(\sqrt{m_r}\right)^n e^{-in\theta_r} \right)$$
$$\times e^{-\nu\left(\sum_{r=1}^n m_r^R\right)+\left(R\nu-n+\frac{R-1}{2}\right)\left(\sum_{r=1}^n \log m_r\right)} \tag{4.13}$$

In this case also we can avoid tedious calculation by applying the permutation trick. The contribution to the first term from the Vandermonde is same as before, and the contribution from $\mathrm{tr}Z^n\,\mathrm{tr}Z^{\dagger n}$ is $\sum_{r=1}^n m_r^n$. The net contribution is then $\left(\sum_{r=1}^n m_r^n\right) m_1^{n-1} m_2^{n-2} \cdots m_{n-1}$. Proceeding as before we have after the θ integrals

$$\begin{aligned}
\partial_{-n}\partial_n \mathcal{Z}_{\mathrm{NMM}}^{N=n}(t=0) &= (2\pi)^n n! \int_0^\infty \prod_{r=1}^n dm_r \left(\sum_{r=1}^n m_r^n\right) \prod_{j=1}^{n-1} m_j^{n-j} \\
&\qquad \times e^{-\nu\left(\sum_{r=1}^n m_r^R\right)+\left(R\nu-n+\frac{R-1}{2}\right)\left(\sum_{r=1}^n \log m_r\right)} \\
&= (2\pi)^n n! \sum_{j=1}^n \left[\nu^{-\left(\nu+\frac{1}{2}-\left(j-n-\frac{1}{2}\right)\frac{1}{R}\right)} \Gamma\left(\nu+\frac{1}{2}-\left(j-n-\frac{1}{2}\right)\frac{1}{R}\right) \right. \\
&\qquad \left. \times \prod_{\substack{r=1 \\ r\neq j}}^n \nu^{-\left(\nu+\frac{1}{2}-\left(r-\frac{1}{2}\right)\frac{1}{R}\right)} \Gamma\left(\nu+\frac{1}{2}-\left(r-\frac{1}{2}\right)\frac{1}{R}\right) \right]
\end{aligned} \tag{4.14}$$

From eq. (4.12) and eq. (4.14) we find (after changing variables $j \to n+1-r$):

$$\langle T_{-n/R} T_{n/R} \rangle_{\mathrm{NMM}}^{N=n} = \nu^{-n/R} \sum_{r=1}^n \frac{\Gamma\left(\frac{1}{2}-i\mu+\left(r-\frac{1}{2}\right)\frac{1}{R}\right)}{\Gamma\left(\frac{1}{2}-i\mu+\left(r-n-\frac{1}{2}\right)\frac{1}{R}\right)} \tag{4.15}$$

As before, we analytically continue $\nu = -i\mu$ and take the modulus to get:

$$\langle T_{-n/R} T_{n/R} \rangle_{\mathrm{NMM}}^{N=n} = \mu^{-n/R} \left| \sum_{r=1}^n \frac{\Gamma\left(\frac{1}{2}-i\mu+\left(r-\frac{1}{2}\right)\frac{1}{R}\right)}{\Gamma\left(\frac{1}{2}-i\mu+\left(r-n-\frac{1}{2}\right)\frac{1}{R}\right)} \right| \tag{4.16}$$

After changing normalisation via eq. (4.6), we see that this agrees perfectly with eq. (3.8).

The above calculation was performed with matrices of rank $N = n$. It can easily be repeated for the other cases. When N is smaller than n, we find that the answer, as a function of μ, is not equal to the correct two-point function, and does not become the correct one even after choosing $\mu = in/R$. As before, this is due to "contamination" by correlators of higher point functions carrying the same total momentum, because for $N < n$ the corresponding correlators in the NMM are not all linearly independent. For $N > n$, instead, we actually get the *same* final answer as for $N = n$. The computational procedure we described above seems to suggest that extra terms arise for $N > n$, but actually they are cancelled by contributions from the θ dependent terms in the Vandermonde factor. Thus when we take the ratio of $\partial_{-n}\partial_n\mathcal{Z}$ and $\mathcal{Z}$ we end up with the r.h.s. of eq. (4.15). Therefore as long as we take $N \geq n$, we get the right answer (independent of N) for every N. This is what we referred to as "stabilisation" above.

4.3 Four-point functions

Now we would like to compute the four-point function in the Normal Matrix Model. For $N = 1$, the calculation has already been performed, since as we noted above, it is the same as the corresponding calculation for the two-point function in eq. (4.8) (more precisely the disconnected four-point function is the same as this two-point function). As we explained there, the result so obtained is a linear combination of the correct two, three and four-point functions of the $c = 1$ string, and to distinguish them we need to go to a higher value of N. Accordingly we have computed the above four-point function using the $N = 2$ NMM. The derivation can be found in appendix C, and the result is:

$$\langle (\mathcal{T}_{-1/R}\mathcal{T}_{1/R})^2 \rangle_{\text{NMM}}^{N=2} = \mu^{-2/R} \left| \mathcal{F}^{+}\left(\tfrac{3}{2R}, -\tfrac{1}{2R}; \mu\right) - \tfrac{1}{2}\Big(\mathcal{F}^{+}\left(\tfrac{1}{2R}, -\tfrac{1}{2R}; \mu\right) \Big)^2 \right| \tag{4.17}$$

Changing from MQM to NMM normalisation using eq. (4.6), and inserting the usual $1/R$ factor, we see that eq. (4.17) above is identical to eq. (3.16).

For completeness, let us briefly consider the two three-point functions

$$\langle \mathcal{T}_{-2R}\mathcal{T}_{R}\mathcal{T}_{R} \rangle_{\text{NMM}}^{N=2}, \qquad \langle \mathcal{T}_{2R}\mathcal{T}_{-R}\mathcal{T}_{-R} \rangle_{\text{NMM}}^{N=2} \tag{4.18}$$

The two are actually equal to each other because of the symmetry $X \to -X$, where X is the Euclidean time direction. We have calculated these correlators both from MQM and NMM (at $N = 2$) and the agreement is exactly as for the cases considered above.

4.4 Why it works

As we reviewed in section 2, the Normal Matrix Model determines every momentum correlator by differentiation with respect to the momentum couplings $t, \bar{t}$. However, the correlators so obtained should only be correct in the limit $N \to \infty$ ("Model I", eq. (2.14)) or the special values $N = \nu R$ ("Model II", eq. (2.15)). Now in the previous subsections we have shown in several examples (including the infinite set of two-point functions) that, given the total momentum P flowing in the correlator, the NMM with matrices of any rank $N \geq P$ suffices to compute the correlator *completely* as a function of μ and R. In view of this, the NMM appears to go beyond its expected range of validity. Here we will give an explanation as to how this comes about.

The basic observation is that the phenomenon we are observing is not to be viewed as an application of Model II, but rather of Model I. Indeed, using Model II and a definite value of N, it is clear from eq. (2.15) that the answers obtained are correct only for a definite value of ν, namely $\nu = N/R$. This relation between N and ν defines a line in (N, ν) space, and the points on this line where N takes integer values are the ones where the procedure works. However, it is clear that in this way one can never recover the full ν dependence at a fixed N.

In contrast, in Model I one is supposed to compute correlators at an arbitrarily large value of N and in the limit $N \to \infty$, the correct answers are obtained as a function of ν. What we will now show is that, after computing a given correlator of total momentum P in this way, and then dividing by the partition function, infinitely many terms cancel out exactly in the ratio. The remaining terms, which actually contribute to the correlator of interest, are the same as one would compute for a finite value of N, namely $N = P$.

The argument goes as follows. From the derivation we have given in the previous subsections and the appendices, any correlator is generated (after θ_i integrations) by inserting an expression of the form $\prod_{i=1}^{N} m_i^{\alpha_i}$ into the m_i integrals, where $\{\alpha_i\}$ correspond to ordered partitions of P. Therefore we should first of all choose N large enough so that all such partitions can be realised and are distinguishable. This is possible for $N \geq P$. For $N < P$ we will miss some partitions, and thus the answer cannot be correct. But the case $N > P$ realises the same partitions as the case $N = P$ and thus gives the same answer. This causes what we earlier called "stabilisation", which amounts to saying that the result for $N = P$ is identical to the result for any $N > P$, and therefore for $N = \infty$. Invoking the converse of stabilisation, we can therefore start with the model defined at $N = \infty$ and "bring back" the value of N to any finite value $N \geq P$ without changing the result. This explains why a finite-N matrix model is sufficient to compute any momentum correlator.

4.5 Combinatorial result for $2n$-point functions

We have shown that the NMM is an effective tool by re-computing known correlators. Now that we understand how and why it works, we apply it to compute a new result: the full (connected plus disconnected) $2n$-point function $\langle (T_{-1/R}T_{1/R})^n\rangle$ for every n and to all orders in perturbation theory. The result, derived in appendix D, is the following:

$$\langle (T_{-1/R}T_{1/R})^n\rangle = \left| \sum_{\{k_i\}} C(\{k_i\})^2 \prod_{i=1}^{n} \frac{\Gamma\left(\frac{1}{2} - i\mu + (k_i - n + \frac{1}{2})\frac{1}{R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - (i - \frac{1}{2})\frac{1}{R}\right)} \right| \tag{4.19}$$

with $C(\{k_i\})$ defined as:

$$C(\{k_i\}) = \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \prod_{i=1}^{n} \binom{n - \sum_{j=1}^{i-1}(k_j - \mathcal{P}_j)}{k_i - \mathcal{P}_i} \tag{4.20}$$

Here, $\{k_i\}$ are strictly ordered partitions of $n(n+1)/2$, namely:

$$k_1 > k_2 > \cdots > k_n, \qquad \sum_{i=1}^{n} k_i = \frac{n(n+1)}{2} \tag{4.21}$$

and $\mathcal{P}$ denote permutations of the n numbers $n-1, n-2, \ldots, 0$.

Let us examine this result more closely. In principle, for every n the answer is a sum of terms, each one being the ratio of n Γ-functions divided by n Γ-functions. However in practice, some of the numerator and denominator terms can cancel out. We can see this more explicitly if we list the first few special cases, of which the first two have already been noted above:

$$\begin{aligned}
\langle T_{-1/R}T_{1/R}\rangle &= \left| \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{1}{2R}\right)} \right| \\
\langle (T_{-1/R}T_{1/R})^2\rangle &= \left| \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{3}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{1}{2R}\right)} + \frac{\Gamma\left(\frac{1}{2} - i\mu + \frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - \frac{3}{2R}\right)} \right| \\
\langle (T_{-1/R}T_{1/R})^3\rangle &= \left| \frac{\Gamma(\frac{1}{2} - i\mu + \frac{5}{2R})}{\Gamma(\frac{1}{2} - i\mu - \frac{1}{2R})} + 4\frac{\Gamma(\frac{1}{2} - i\mu + \frac{3}{2R})}{\Gamma(\frac{1}{2} - i\mu - \frac{3}{2R})} + \frac{\Gamma(\frac{1}{2} - i\mu + \frac{1}{2R})}{\Gamma(\frac{1}{2} - i\mu - \frac{5}{2R})} \right|
\end{aligned} \tag{4.22}$$

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The pattern emerging so far is misleadingly simple, as we see with the next example, the 8-point function:

$$\langle (T_{-1/R}T_{1/R})^4\rangle = \left| \frac{\Gamma(\frac{1}{2}-i\mu+\frac{7}{2R})}{\Gamma(\frac{1}{2}-i\mu-\frac{1}{2R})} + 9\,\frac{\Gamma(\frac{1}{2}-i\mu+\frac{5}{2R})}{\Gamma(\frac{1}{2}-i\mu-\frac{3}{2R})} + 9\,\frac{\Gamma(\frac{1}{2}-i\mu+\frac{3}{2R})}{\Gamma(\frac{1}{2}-i\mu-\frac{5}{2R})} \right.$$
$$\left. + \frac{\Gamma(\frac{1}{2}-i\mu+\frac{1}{2R})}{\Gamma(\frac{1}{2}-i\mu-\frac{7}{2R})} + 4\,\frac{\Gamma(\frac{1}{2}-i\mu+\frac{1}{2R})}{\Gamma(\frac{1}{2}-i\mu-\frac{1}{2R})}\frac{\Gamma(\frac{1}{2}-i\mu+\frac{3}{2R})}{\Gamma(\frac{1}{2}-i\mu-\frac{3}{2R})}\right| \tag{4.23}$$

We see that as the number of operators in the correlator grows, one gets products of more and more Γ-functions in the numerator and denominator. In this example we also see clearly that the coefficients are perfect squares.

Ideally one would like to know the connected part of the $2n$-point function. In principle this can of course be obtained by repeated application of eq. (4.19), but one would like a more explicit and useful expression. However, for the most likely application, to the vortex condensate, we will not really need to make the distinction between connected and disconnected correlators. The vortex condensate corresponds to the partition function of a perturbed theory, and to find the connected component of that it suffices to take a logarithm. We will discuss this issue further in the following section.

5. Applications

5.1 T-duality at $c = 1$

In this subsection we discuss how our results can be applied to check T-duality of the $c = 1$ matrix model. As we have seen, in the Euclidean (finite-temperature) MQM, the momentum and winding modes with respect to the time direction are independently defined. The former arise from macroscopic loops defined in terms of fermion bilinears, while the latter are Wilson-Polyakov loops in the thermal direction, which project the theory onto nonsinglet sectors. From the continuum description we expect that there should be T-duality between these two sets of observables. Indeed, in ref. [16] it has been formally argued that, like the momentum-perturbed matrix model, the winding-perturbed MQM also corresponds to the τ-function of a Toda hierarchy. To understand T-duality better, one would like to compare explicit correlation functions computed from the momentum and winding sides.

An attempt to directly check T-duality was made by Maldacena in [19], where the following two quantities were compared: (i) the two-point function of unit-momentum tachyons, after T-duality, and (ii) the partition function of MQM in the adjoint sector. From eq. (3.11) we see that (i) is equal to:

$$\langle \mathcal{T}_{-R}\mathcal{T}_R\rangle = \left|\frac{\Gamma\left(\frac{1}{2}-i\mu R+\frac{R}{2}\right)}{\Gamma\left(\frac{1}{2}-i\mu R-\frac{R}{2}\right)}\right| \tag{5.1}$$

However, at this point we recall that leg-pole factors of $\Gamma(-|q|)$ were dropped after eq. (3.1). Restoring them and taking the large-μ asymptotics of this correlator, we find:[8]

$$\langle \mathcal{T}_{-R}\mathcal{T}_R\rangle = \big(\Gamma(-R)\big)^2(\mu R)^R\left(1+\frac{1}{24}\Big(R-\frac{1}{R}\Big)\mu^{-2}+\mathcal{O}\left(\mu^{-4}\right)\right) \tag{5.2}$$

On the other hand, (ii) is obtained by solving MQM in the adjoint sector. In the large N limit, Maldacena obtained the leading (tree level) contribution to the partition function in this sector as:

$$\frac{Z_{\text{adj}}}{Z_{\text{sing}}} = \langle \mathcal{W}_{\text{adj}}\rangle = \frac{1}{4\sin^2\pi R}\,\mu^R = \frac{1}{4\pi^2}\left(\Gamma(R+1)\Gamma(-R)\right)^2\mu^R \tag{5.3}$$

The power of μ agrees with that in the leading term of eq. (5.2). The remaining discrepancy can be assigned to the normalisation of the fundamental Wilson-Polyakov loop (or equivalently to the normalisation of the original momentum modes), and we see that eqs. (5.3) and (5.2) agree to leading order if we change the normalisation of this loop variable to:

$$\mathcal{W}_N \to \frac{1}{2\pi}\,\frac{R^{\frac{R}{2}}}{\Gamma(R+1)}\mathcal{W}_N \tag{5.4}$$

This is a relatively simple change of normalisation,[9] and appears to specify the basis in which T-duality holds in MQM.

It is not entirely surprising that one needs to change normalisation of the matrix model observables in order to implement T-duality. Indeed, this duality is most manifest in the worldsheet or Liouville approach, in which the momentum and winding vertex operators come with a natural normalisation and are related to each other by the simple change $(X_L, X_R) \to (X_L, -X_R)$. On the matrix model side, momentum operators in the MQM are related to the corresponding Liouville operators by a change of normalisation, eq. (2.4). So one should expect that winding operators in MQM are also related to Liouville winding operators by a change of normalisation.

This is not to say we understand the nature of these normalisation factors in general. In fact, as stressed in ref. [19], we need more examples in order to check the consistency of this picture. As an example, if one could compute the genus-1 correction to the adjoint sector partition function, this could be compared with the genus-1 term in eq. (5.2). Similarly, if one could compute the leading term for those higher representations that correspond to $2n$-point functions of the winding tachyon, then one could match this with the asymptotics of the latter, which can be read off from our results in section 4.5.

There will also be representations corresponding to the correlators of multiply wound tachyons $\mathcal{T}_{nR}$. These correlators can be found by T-dualising the relevant momentum correlators, for example the two-point functions are found by T-dualising eq. (3.8), leading to:

$$\langle \mathcal{T}_{-nR}\mathcal{T}_{nR}\rangle \;=\; \big(\Gamma(-nR)\big)^2\left|\sum_{r=1}^{n}\frac{\Gamma(\frac{1}{2}-i\mu R+(r-\frac{1}{2})R)}{\Gamma(\frac{1}{2}-i\mu R+(r-n-\frac{1}{2})R)}\right| \tag{5.5}$$

[8] The factor R^R was not written in ref. [19].

[9] Notice that the normalisation factor becomes trivial at the special radius $R = 1$.

$$= n(\mu R)^{nR}\big(\Gamma(-nR)\big)^2\left(1 - \frac{nR(nR-1)\big((n^2-1)R^2 - nR - 1\big)}{24R^2}\mu^{-2} + \mathcal{O}\left(\mu^{-4}\right)\right)$$

In the matrix model, this should correspond to the fundamental Wilson-Polyakov loop with a contour that winds n times over the time direction. In principle we are allowed an independent choice of normalisation for each winding number. In fact the momentum and winding modes have corresponding freedoms in normalisation, and the only thing relevant for T-duality is the relative normalisation between them. So when we consider the nonsinglet sector related to multiply wound loops, and the corresponding tachyons of n units of momentum, the leading-order comparison will be used to fix the normalisation and the loop corrections will constitute a genuine check of T-duality.

To summarise, we have not been able to address the problem of T-duality but only set up one side of it. Namely, we have exhibited the all-orders finite-radius correlators computed from the momentum side, after performing a T-duality transformation. This constitutes a prediction to be checked once it is properly understood how to perform nonsinglet computations for different representations and to higher orders in string perturbation theory.

There is one more intriguing point that we would like to mention. The correlators we have computed take very special values at the selfdual radius $R = 1$, the point of enhanced SU(2) symmetry. In particular, all loop corrections to the two-point function of unit momentum tachyons vanish, as can be seen from eq. (3.11). Thus the tree level answer is exact.[10] By T-duality the same property should hold for the two-point function of unit winding modes. It is plausible that one could extract this simple property just from the structure of the nonsinglet Hamiltonian — in this case it is the adjoint Hamiltonian that was studied in ref. [19], specialised to $R = 1$. Similarly, at $R = 1$ the other two-point functions have perturbation series that terminate at a finite number of loops, as one can easily check from eq. (5.5). So, for consistency this must also be a property of the antisymmetric-antisymmetric representations referred to above. It may be simpler to derive this kind of general result in the nonsinglet sector than to actually compute coefficients with precision.

5.2 Vortex condensate and black holes

It is believed that the Euclidean 2D black hole background, defined in the continuum by an $\mathrm{SL}(2,R/\,\mathrm{U}(1)$ CFT, is equivalent to the $c = 1$ matrix model perturbed by fundamental Wilson-Polyakov loops:

$$S_{\mathrm{MQM}} \to S_{\mathrm{MQM}} + \lambda\,\mathcal{W}_N + \bar{\lambda}\,\mathcal{W}_{\bar{N}} \tag{5.6}$$

The basis for this belief is the FZZ conjecture [15], which relates the black hole background to Sine-Liouville theory.[11] Via the equivalence in eq. (2.8), the latter is the same as the perturbed background above.

[10] This was already known long ago, for example as the puncture equation in the Kontsevich-Penner model [22].

[11] This conjecture has been proved by Hori and Kapustin [33] in the $\mathcal{N} = 2$ supersymmetric case. As Maldacena has argued [19], suitably orbifolding both sides of their argument leads to a proof for the bosonic case.

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To be precise, the FZZ conjecture is not really an either/or statement wherein one uses either the black hole background or the Sine-Liouville perturbation. It has increasingly become clear that the backgrounds that one might call "black hole" or "Sine-Liouville" are the same, and both perturbations are turned on simultaneously. Depending on the value of the worldsheet coupling, one or the other of these perturbations is more dominant, but for example the exact correlation functions have poles corresponding to both perturbations.[12] In the present work we will not focus on these details, but will be content to treat the black hole story as a motivation to understand the vortex condensate:

$$\langle e^{\lambda\,\mathcal{W}_N+\bar\lambda\,\mathcal{W}_{\bar N}}\rangle\Big|_{\rm MQM} \tag{5.7}$$

One way to compute this condensate would be to sum over an infinite set of nonsinglet sectors in the MQM with some definite weights. However, as we have seen, the technology to do this seems rather limited at present. An alternative is to assume T-duality to compute the correlator:

$$\langle e^{\lambda\,\mathcal{T}_R+\bar\lambda\,\mathcal{T}_{-R}}\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\lambda^n}{n!}\frac{\bar\lambda^m}{m!}\langle(\mathcal{T}_R)^n(\mathcal{T}_{-R})^m\rangle = \sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{(n!)^2}\langle(\mathcal{T}_R\mathcal{T}_{-R})^n\rangle \tag{5.8}$$

where the last equality follows from conservation of winding number.

Now from the computation in appendix D, we have the following result after T-duality:

$$\langle(\mathcal{T}_{-R}\mathcal{T}_R)^n\rangle = \left|\sum_{\{k_i\}}C(\{k_i\})^2\prod_{i=1}^{n}\frac{\Gamma\left(\frac12 - i\mu R-(i+k_i-\frac12)R\right)}{\Gamma\left(\frac12 - i\mu R-(i-\frac12)R\right)}\right| \tag{5.9}$$

where $\{k_i\}$ are strictly ordered partitions of $n(n+1)/2$, and $C(\{k_i\})$ are the combinatorial coefficients given in eq. (D.9).

The above correlators contain both connected and disconnected contributions. We can now pass to the generating function:

$$\langle e^{\lambda\,\mathcal{T}_R+\bar\lambda\,\mathcal{T}_{-R}}\rangle = \sum_{n=0}^{\infty}\frac{|\lambda|^{2n}}{(n!)^2}\left|\sum_{\{k_i\}}C(\{k_i\})^2\prod_{i=1}^{n}\frac{\Gamma\left(\frac12 - i\mu R-(i+k_i-\frac12)R\right)}{\Gamma\left(\frac12 - i\mu R-(i-\frac12)R\right)}\right| \tag{5.10}$$

This is the partition function in the presence of a vortex condensate, and its logarithm is the free energy of the perturbed theory. So one does not need at any point to compute individual connected correlators.

The above expression is completely explicit and does not require integrating any equation or developing a recursion relation. We expect it will be useful to to extract physical quantities of interest related to the Euclidean 2d black hole. This is beyond the scope of the present work, however, and we hope to return to a more detailed analysis of this formula in the future.

Again it is worth pointing out that at the selfdual radius $R = 1$ the vortex condensate is known exactly, though deriving it from the above expression would not be the easiest

[12] See for example ref. [6]. We are grateful to Ari Pakman for explaining this to us.

way. The puncture equation of ref. [22] simply tells us that:

$$\langle e^{\lambda\,\mathcal{T}_R+\bar{\lambda}\,\mathcal{T}_{-R}}\rangle|_{R=1} = e^{\mu\lambda\bar{\lambda}} \tag{5.11}$$

and one can check easily that this agrees with the cases in eq. (4.22) specialised to $R = 1$.

The significance for the Euclidean 2d black hole of this simple result has not, to our knowledge, been explored. While it is true that the black hole CFT corresponds to a radius $R = \frac{3}{2}$, it is believed [16] to have a marginal deformation to other radii at least in the range $1 < R < 2$. So the physical consequences of the simple formula above at $R = 1$ would be worth understanding better.

6. Conclusions

In this work we have examined the familiar $c = 1$ bosonic noncritical string theory, or rather its Euclidean (finite temperature) version, from the perspective of correlation functions. Both old and new techniques were used to develop simple, elegant and explicit formulae as functions of two variables: the cosmological constant μ and the compactification radius or inverse temperature R. The key results are summarised in eqs. (3.8), (3.16), (4.19). In addition we have shown that the Normal Matrix Model is a powerful computational tool.

An obvious extension of this work would be to the case of noncritical type 0 strings [34, 35]. In ref. [36], explicit expressions are obtained for the partition functions of type 0A and 0B strings in the presence of fluxes. These expressions are richer than the corresponding ones for the bosonic noncritical string, both because of the flux dependence and because they are nonperturbative in μ. Our work should generalise quite straightforwardly, particularly to the Euclidean type 0B case, and the correlators so obtained will contain nonperturbative information about the theory.

A detailed investigation into the physical questions that motivated the present exercise, namely a better understanding of the 2d black hole background as well as of T-duality in the matrix model, is left for subsequent work. We also note that the physical origin of the Normal Matrix Model has not yet been understood. As it is clearly a correct and useful description of the $c = 1$ string, and moreover makes sense only in the Euclidean context, it would be worth trying to put it on a similar footing as MQM in terms of the dynamics of some appropriate (Euclidean) D-branes.

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A. Computation of two-point functions in the NMM

Here we present some of the details of how to compute two-point functions in the Normal Matrix Model. To start with, for the partition function we have

$$\mathcal{Z}_{\rm NMM}^{N=2}(t=0) = \int d^2z_1 d^2z_2\, |z_1-z_2|^2 \times e^{-\nu\left((z_1\bar z_1)^R+(z_2\bar z_2)^R\right)+\left(R\nu-2+\frac{R-1}{2}\right)(\log z_1\bar z_1+\log z_2\bar z_2)} \quad \text{(A.1)}$$

As before, we change variables $z_i=\sqrt{m_i}\,e^{i\theta_i}$, $d^2z_i\to dm_i\,d\theta_i$ and we get

$$\begin{aligned}\mathcal{Z}_{\rm NMM}^{N=2}(t=0) &= \int_0^\infty dm_1dm_2\int_0^{2\pi}d\theta_1d\theta_2\,\left(m_1+m_2-\sqrt{m_1m_2}(e^{i\theta_{12}}+e^{-i\theta_{12}})\right)\\ &\quad\times e^{-\nu(m_1^R+m_2^R)+\left(R\nu-2+\frac{R-1}{2}\right)(\log m_1+\log m_2)}\\ &= 4\pi^2\int_0^\infty dm_1dm_2\,(m_1+m_2)(m_1m_2)^{\left(R\nu-2+\frac{R-1}{2}\right)}e^{-\nu\left(m_1^R+m_2^R\right)}\\ &= \tfrac{8\pi^2}{R^2}\nu^{-\left(\nu+\frac12-\frac{1}{2R}\right)}\Gamma\left(\nu+\tfrac12-\tfrac{1}{2R}\right)\times\nu^{-\left(\nu+\frac12-\frac{3}{2R}\right)}\Gamma\left(\nu+\tfrac12-\tfrac{3}{2R}\right),\end{aligned} \quad \text{(A.2)}$$

where $\theta_{12}\equiv\theta_1-\theta_2$. In a similar manner we have

$$\begin{aligned}\partial_{-2}\partial_2\mathcal{Z}_{\rm NMM}^{N=2}(t=0) &= \int d^2z_1d^2z_2\,|z_1-z_2|^2(z_1^2+z_2^2)(\bar z_1^2+\bar z_2^2)\\ &\quad\times e^{-\nu\left((z_1\bar z_1)^R+(z_2\bar z_2)^R\right)+\left(R\nu-2+\frac{R-1}{2}\right)(\log z_1\bar z_1+\log z_2\bar z_2)}\\ &= \int_0^\infty dm_1dm_2\int_0^{2\pi}d\theta_1d\theta_2\,\left(m_1+m_2-\sqrt{m_1m_2}(e^{i\theta_{12}}+e^{-i\theta_{12}})\right)\\ &\quad\times\left(m_1^2+m_2^2+m_1m_2e^{2i\theta_{12}}+m_1m_2e^{-2i\theta_{12}}\right)e^{-\nu(m_1^R+m_2^R)+\left(R\nu-2+\frac{R-1}{2}\right)(\log m_1+\log m_2)}\\ &= 4\pi^2\int_0^\infty dm_1dm_2\,(m_1+m_2)(m_1^2+m_2^2)(m_1m_2)^{\left(R\nu-2+\frac{R-1}{2}\right)}e^{-\nu\left(m_1^R+m_2^R\right)}\end{aligned}$$

Evaluating the integrals on m_1,m_2 we get

$$\begin{aligned}\partial_{-2}\partial_2\mathcal{Z}_{\rm NMM}^{N=2}(t=0) =\ & \tfrac{8\pi^2}{R^2}\nu^{-\left(\nu+\frac12+\frac{3}{2R}\right)}\Gamma\left(\nu+\tfrac12+\tfrac{3}{2R}\right)\times\nu^{-\left(\nu+\frac12-\frac{3}{2R}\right)}\Gamma\left(\nu+\tfrac12-\tfrac{3}{2R}\right)\\ &+\tfrac{8\pi^2}{R^2}\nu^{-\left(\nu+\frac12+\frac{1}{2R}\right)}\Gamma\left(\nu+\tfrac12+\tfrac{1}{2R}\right)\times\nu^{-\left(\nu+\frac12-\frac{1}{2R}\right)}\Gamma\left(\nu+\tfrac12-\tfrac{1}{2R}\right)\end{aligned} \quad \text{(A.3)}$$

From eq. (A.2) and eq. (A.3) we have

$$\langle T_{-2/R}T_{2/R}\rangle_{\rm NMM}^{N=2} = \nu^{-2/R}\left(\frac{\Gamma\left(\nu+\frac12+\frac{3}{2R}\right)}{\Gamma\left(\nu+\frac12-\frac{1}{2R}\right)}+\frac{\Gamma\left(\nu+\frac12+\frac{1}{2R}\right)}{\Gamma\left(\nu+\frac12-\frac{3}{2R}\right)}\right) \quad \text{(A.4)}$$

As before, to get the correct two point function we have to analytically continue $\nu=-i\mu$ and take the modulus of the above expression. This gives:

$$\langle T_{-2/R}T_{2/R}\rangle_{\rm NMM}^{N=2} = \mu^{-2/R}\left|\frac{\Gamma\left(\frac12-i\mu+\frac{3}{2R}\right)}{\Gamma\left(\frac12-i\mu-\frac{1}{2R}\right)}+\frac{\Gamma\left(\frac12-i\mu+\frac{1}{2R}\right)}{\Gamma\left(\frac12-i\mu-\frac{3}{2R}\right)}\right| \quad \text{(A.5)}$$

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B. Evaluation of a summation in the MQM four-point function

In order to show the equivalence between eqs. (3.15) and (3.16) we need to prove the following identity:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\Gamma(-\frac{1}{R}+n)}{\Gamma(-\frac{1}{R})}\right)^2\frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{3}{2R}-n\right)}{\Gamma\left(\frac{1}{2}-i\mu-\frac{1}{2R}\right)}=\left(\frac{\Gamma\left(\frac{1}{2}-i\mu+\frac{1}{2R}\right)}{\Gamma\left(\frac{1}{2}-i\mu-\frac{1}{2R}\right)}\right)^2 \tag{B.1}$$

Let us start with the expression:

$$\mathcal{E}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\Gamma(-\frac{1}{R}+n)}{\Gamma(-\frac{1}{R})}\right)^2\Gamma\left(\frac{1}{2}-i\mu+\frac{3}{2R}-n\right) \tag{B.2}$$

Using the integral representation of the Γ function we write this as:

$$\mathcal{E}=\frac{1}{\left(\Gamma(-\frac{1}{R})\right)^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int d^3t\ (t_1t_2)^{-\frac{1}{R}+n-1}t_3^{-n+\frac{1}{2}-i\mu+\frac{3}{2R}-1}e^{-t_1-t_2-t_3} \tag{B.3}$$

The sum over n can now be performed immediately and we have:

$$\begin{aligned}\mathcal{E}&=\frac{1}{\left(\Gamma(-\frac{1}{R})\right)^2}\int d^3t\ e^{-\frac{t_1t_2}{t_3}}(t_1t_2)^{-\frac{1}{R}-1}\,t_3^{\frac{1}{2}-i\mu+\frac{3}{2R}-1}\,e^{-t_1-t_2-t_3}\\&=\frac{1}{\left(\Gamma(-\frac{1}{R})\right)^2}\int d^3t\ e^{-t_1(1+\frac{t_2}{t_3})}(t_1t_2)^{-\frac{1}{R}-1}\,t_3^{\frac{1}{2}-i\mu+\frac{3}{2R}-1}\,e^{-t_2-t_3}\end{aligned} \tag{B.4}$$

Using the change of variables $t_1\to t_1(1+\frac{t_2}{t_3})$ and performing the integral on t_1 we get:

$$\mathcal{E}=\frac{1}{\Gamma(-\frac{1}{R})}\int d^2t\ t_2^{-\frac{1}{R}-1}\,t_3^{\frac{1}{2}-i\mu+\frac{3}{2R}-1}\,t_3^{-\frac{1}{R}}\,(t_2+t_3)^{\frac{1}{R}}\,e^{-t_2-t_3} \tag{B.5}$$

We next introduce a parameter α which allows us to write the above equation as:

$$\mathcal{E}=\frac{1}{\Gamma(-\frac{1}{R})}\left(-\frac{\partial}{\partial\alpha}\right)^{\frac{1}{R}}\int d^2t\ t_2^{-\frac{1}{R}-1}\,t_3^{\frac{1}{2}-i\mu+\frac{1}{2R}-1}\,e^{-\alpha(t_2+t_3)}\Big|_{\alpha=1} \tag{B.6}$$

Changing variables $t_i\to\alpha t_i$ we have:

$$\begin{aligned}\mathcal{E}&=\frac{1}{\Gamma(-\frac{1}{R})}\left(-\frac{\partial}{\partial\alpha}\right)^{\frac{1}{R}}\alpha^{-\frac{1}{2}+i\mu+\frac{1}{2R}}\Big|_{\alpha=1}\int dt_2\ t_2^{-\frac{1}{R}-1}\,e^{-t_2}\int dt_3\ t_3^{\frac{1}{2}-i\mu+\frac{1}{2R}-1}\,e^{-t_3}\\&=\Gamma(\tfrac{1}{2}-i\mu+\tfrac{1}{2R})\left(-\frac{\partial}{\partial\alpha}\right)^{\frac{1}{R}}\alpha^{-\frac{1}{2}+i\mu+\frac{1}{2R}}\Big|_{\alpha=1}\end{aligned} \tag{B.7}$$

Using the relation:

$$\left(-\frac{\partial}{\partial\alpha}\right)^m\alpha^n\Big|_{\alpha=1}=\frac{\Gamma(-n+m)}{\Gamma(-n)} \tag{B.8}$$

we finally have:

$$\mathcal{E}=\frac{\left(\Gamma(\frac{1}{2}-i\mu+\frac{1}{2R})\right)^2}{\Gamma(\frac{1}{2}-i\mu-\frac{1}{2R})} \tag{B.9}$$

Using eq. (B.2) and dividing both sides by $\Gamma(\frac{1}{2}-i\mu-\frac{1}{2R})$ we immediately get eq. (B.1).

C. Four-point function in NMM

We now briefly describe the calculation of the connected four-point function of unit momentum modes in the NMM. This is obtained by differentiating the free energy $\mathcal{F}$ with respect to the couplings. We thus have:

$$\begin{aligned}\langle (T_{-1/R}T_{1/R})^2\rangle &= \partial_{-1}^2\partial_1^2\mathcal{F} \\ &= \langle (T_{-1/R}T_{1/R})^2\rangle^{\rm disconn} - 2\langle T_{-1/R}T_{1/R}\rangle^2\end{aligned} \tag{C.1}$$

where $\mathcal{F} = \ln \mathcal{Z}_{\rm NMM}$. The second term in the above equation can be calculated from the NMM with $N = 2$ and is given by:

$$\begin{aligned}\langle (T_{-1/R}T_{1/R})^2\rangle^{\rm disconn} &= \langle ({\rm tr}Z^\dagger)^2({\rm tr}Z)^2\rangle^{N=2}_{\rm NMM} \\ &= \nu^{-2/R}\left(\frac{\Gamma\left(\nu+\frac{1}{2}+\frac{3}{2R}\right)}{\Gamma\left(\nu+\frac{1}{2}-\frac{1}{2R}\right)}+\frac{\Gamma\left(\nu+\frac{1}{2}+\frac{1}{2R}\right)}{\Gamma\left(\nu+\frac{1}{2}-\frac{3}{2R}\right)}\right)\end{aligned}$$

The explicit calculation is very similar to the calculation of $\langle T_{-2/R}T_{2/R}\rangle$ from the NMM. The disconnected piece is simply the square of the two-point function listed in eq. (4.4). Putting everything together the connected four-point function is given by:

$$\begin{aligned}\langle (T_{-1/R}T_{1/R})^2\rangle^{\rm conn}_{\rm NMM} = \nu^{-2/R}\Bigg[&\frac{\Gamma\left(\nu+\frac{1}{2}+\frac{3}{2R}\right)}{\Gamma\left(\nu+\frac{1}{2}-\frac{1}{2R}\right)}+\frac{\Gamma\left(\nu+\frac{1}{2}+\frac{1}{2R}\right)}{\Gamma\left(\nu+\frac{1}{2}-\frac{3}{2R}\right)} \\ &-2\left(\frac{\Gamma(n+\frac{1}{2}+\frac{1}{2R})}{\Gamma(n+\frac{1}{2}+\frac{1}{2R})}\right)^2\Bigg]\end{aligned} \tag{C.2}$$

Analytically continuing $\nu = -i\mu$ and taking the modulus, and then using the definition of $\mathcal{F}^+$ in eq. (3.4), we finally get:

$$\langle (T_{-1/R}T_{1/R})^2\rangle^{\rm conn}_{\rm NMM} = (\mu)^{-2/R}\left|\mathcal{F}^+(\tfrac{3}{2R},-\tfrac{1}{2R};\mu)-\tfrac{1}{2}\Big(\mathcal{F}^+(\tfrac{1}{2R},-\tfrac{1}{2R};\mu)\Big)^2\right| \tag{C.3}$$

D. $2n$-point functions in NMM

Here we present the detailed calculation of the $2n$-point functions from the NMM. In what follows we will take the rank of the matrix, N, to be equal to n. We have:

$$\begin{aligned}(\partial_{-1}\partial_1)^n\mathcal{Z}^{N=n}_{\rm NMM}(t=0) = &\int\prod_{i=1}^n d^2z_i\prod_{i<j}^n|z_i-z_j|^2\left(\sum_{i=1}^n z_i\right)^n\left(\sum_{i=1}^n \bar{z}_i\right)^n \\ &\times e^{-\nu\sum_{r=1}^n(z_r\bar{z}_r)^R+\left(R\nu-n+\frac{R-1}{2}\right)\sum_{r=1}^n\log z_r\bar{z}_r}\end{aligned} \tag{D.1}$$

We would now like to make the substitution $z_i = \sqrt{m_i}\,e^{i\theta_i}$ and perform the θ integrals. The remaining integrand will then be a function of the m_i and we will find that it has the form $\left(\sum_{\{k_i\}} C(\{k_i\})^2\prod_i m_i^{k_i} + \text{permutations}\right)e^{-S_{\rm NMM}}$. Here $\{k_i\}$ are positive integers corresponding to *strictly ordered* partitions of $n(n+1)/2$, i.e.:

$$\sum_{i=1}^n k_i = \tfrac{n(n+1)}{2}, \quad k_1 > k_2 > \cdots > k_n \geq 0 \tag{D.2}$$

The permutations referred to are of the m_i. Because the m_i are integration variables, summing over permutations simply amounts to multiplying by a factor of $n!$. The constant coefficients have been labelled $C(\{k_i\})^2$ in anticipation of the fact that they will turn out to be squares. After performing the integration over m_i and dividing by $Z_{\rm NMM}$ we get the final answer as a sum of ratios of products of gamma functions, with each term in the sum corresponding to a strictly ordered partition $\{k_i\}$ of $n(n+1)/2$.

We will first show that the coefficients are perfect squares $C(\{k_i\})^2$. After that we will turn to the calculation of the $C(\{k_i\})$. Consider the expression:

$$\mathcal{U} = \left(\sum_{i=1}^{n} z_i\right)^n \prod_{j<k}^{n}(z_j - z_k). \tag{D.3}$$

The full integrand is then $\mathcal{U}\bar{\mathcal{U}}$ times the exponential factor. Because the action is independent of the θ's, the entire θ-dependence of the integrand is in $\mathcal{U}\bar{\mathcal{U}}$. Note that $\mathcal{U}$ has only positive powers of $e^{i\theta_i}$ and $\bar{\mathcal{U}}$ has only negative powers. Only the θ-independent terms in the expansion of $\mathcal{U}\bar{\mathcal{U}}$ will survive the θ integrals.

It is easy to see that if we expand $\mathcal{U}$,$\bar{\mathcal{U}}$ then we get:

$$\begin{aligned}\mathcal{U} &= \sum_{\{\alpha_i\}} C(\{k_i\}) \prod_{i=1}^{n} z_i^{k_i} + \text{permutations}\\ \bar{\mathcal{U}} &= \sum_{\{\alpha_i\}} C(\{k_i\}) \prod_{i=1}^{n} \bar{z}_i^{k_i} + \text{permutations},\end{aligned} \tag{D.4}$$

with $\{k_i\}$ defined as before. It is now clear that the coefficients of θ-independent terms in $\mathcal{T}\bar{\mathcal{T}}$ must be perfect squares, as the phase of a term in the first expression of eq. (D.4) can only be cancelled by the complex conjugate term from the second expression, which has the same coefficient as the first term.

Let us now determine the coefficients $C(\{\alpha_i\})$. First we note the following property of the positive phase part of the Vandermonde:

$$\prod_{j<k}^{n}(z_j - z_k) = \sum_{\mathcal{P}}(-1)^{\mathcal{P}} \prod_{j=1}^{n} z_j^{\mathcal{P}_j}, \tag{D.5}$$

where $\mathcal{P}$ is a particular permutation of the n integers $(n-1, n-2, \ldots, 0)$ and $\mathcal{P}_j$ denotes the $j^{\rm th}$ element of the permutation $\mathcal{P}$.[13] The sign for the first permutation is positive by construction. Any other permutation can be arrived at by a series of interchanges $z_i \leftrightarrow z_j$. Each such interchange introduces a minus sign in the Vandermonde. Thus even permutations have a positive sign, while odd permutations have a negative sign, leading to eq. (D.5). Expanding the first factor in eq. (D.3) in a multinomial series and using eq. (D.5) we get:

$$\mathcal{U} = \left(\sum_{\{\beta_i\}} \prod_{i=1}^{n} \binom{n - \sum_{j=1}^{i-1}\beta_j}{\beta_i} z_i^{\beta_i}\right)\left(\sum_{\mathcal{P}}(-1)^{\mathcal{P}} \prod_{j=1}^{n} z_j^{\mathcal{P}_j}\right)$$

[13] For example, $\mathcal{P}_j = n - j$ when $\mathcal{P}$ is the identity permutation.

$$= \sum_{\{\beta_i\}} \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \prod_{i=1}^{n} \binom{n - \sum_{j=1}^{i-1} \beta_j}{\beta_i} z_i^{\beta_i + \mathcal{P}_i} \tag{D.6}$$

where $\{\beta_i\}$ are the *unordered* partitions of n.

Let us examine the possible values of the exponent $k_i = \beta_i + \mathcal{P}_i$ in the above. If $k_i = k_j$ for some $i \neq j$ then the corresponding coefficient is zero. This can be traced back to the fact that the expression eq. (D.3) is odd under pairwise interchange of the z's. Therefore we can rewrite the above as:

$$\mathcal{U} = \sum_{\substack{k_i \neq k_j \\ \sum_i k_i = n(n+1)/2}} \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \prod_{i=1}^{n} \binom{n - \sum_{j=1}^{i-1}(k_j - \mathcal{P}_j)}{k_i - \mathcal{P}_i} \prod_{i=1}^{n} z_i^{k_i} \tag{D.7}$$

Because the k_i are all distinct, we can limit ourselves to strictly ordered sets satisfying $k_1 > k_2 > \cdots k_n$. The other orderings are obtained by permuting these ones, or equivalently by permuting the z_i's. Thus we have:

$$\mathcal{U} = \sum_{\substack{k_1 > k_2 > \cdots > k_n \\ \sum_i k_i = n(n+1)/2}} C(\{k_i\}) \prod_{i=1}^{n} z_i^{k_i} + (\text{permutations of } z_i) \tag{D.8}$$

with

$$C(\{k_i\}) = \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \prod_{i=1}^{n} \binom{n - \sum_{j=1}^{i-1}(k_j - \mathcal{P}_j)}{k_i - \mathcal{P}_i} \tag{D.9}$$

Finally we combine $\mathcal{U}$ with $\bar{\mathcal{U}}$ and integrate over the angles to get:

$$\begin{aligned}(\partial_{-1}\partial_1)^n \mathcal{Z}_{\rm NMM}^{N=n} &= (2\pi)^n \int \prod_{i=1}^{n} dm_i \sum_{\{k_i\}} C(\{k_i\})^2 \prod_{i=1}^{n} m_i^{k_i} e^{\sum_{i=1}^{n}(-\nu m_i^R + (R\nu - n + \frac{R-1}{2}) \log m_i)} \\ &\qquad + \text{permutations} \\ &= (2\pi)^n n! \sum_{\{k_i\}} C(\{k_i\})^2 \prod_{i=1}^{n} \nu^{-\left(\frac{1}{2} + \nu + (k_i - n + \frac{1}{2})\frac{1}{R}\right)} \Gamma\left(\tfrac{1}{2} + \nu + (k_i - n + \tfrac{1}{2})\tfrac{1}{R}\right)\end{aligned} \tag{D.10}$$

Using the expression for the partition function $\mathcal{Z}_{\rm NMM}$ from eq. (4.12) for $N = n$ we have:

$$\frac{(\partial_{-1}\partial_1)^n \mathcal{Z}_{\rm NMM}^{N=n}}{\mathcal{Z}_{\rm NMM}^{N=n}} = \nu^{-n/R} \sum_{\{k_i\}} C(\{k_i\})^2 \prod_{i=1}^{n} \frac{\Gamma\left(\frac{1}{2} + \nu + (k_i - n + \frac{1}{2})\frac{1}{R}\right)}{\Gamma\left(\frac{1}{2} + \nu - (i - \frac{1}{2})\frac{1}{R}\right)} \tag{D.11}$$

The $2n$-point function is given by analytically continuing $\nu = -i\mu$, changing to MQM normalisation using eq. (4.6) (which amounts to removing the power of ν in front), and finally taking the modulus:

$$\langle (T_{-1/R} T_{1/R})^n \rangle = \left| \sum_{\{k_i\}} C(\{k_i\})^2 \prod_{i=1}^{n} \frac{\Gamma\left(\frac{1}{2} - i\mu + (k_i - n + \frac{1}{2})\frac{1}{R}\right)}{\Gamma\left(\frac{1}{2} - i\mu - (i - \frac{1}{2})\frac{1}{R}\right)} \right| \tag{D.12}$$

with $C(\{k_i\})$ given by eq. (D.9).

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